Stability Condition for the Explicit Algorithms of the Time Domain Analysis of Maxwell's Equations

Michał Mrozowski, *Member, IEEE*

Abstract— **This letter presents the derivation of the stability condition for various types of time domain algorithms used in the solution of linear hyperbolic differential equations that arise in the investigation of transient electromagnetic fields. The stability condition of the algorithm is derived by investigating the properties of operators in suitably defined Hilbert spaces. Compared to the classical von Neumann stability analysis, the functional analysis approach gives more general results that can be easily applied to some recent and possible future time domain schemes.**

I. Introduction

EXPLICIT ALGORITHMS for the solution of initial value problems have recently received much attention among researchers involved in the numerical analysis of electromagnetic fields. Two methods belonging to this class, known as finite difference-time domain (FDTD) and transmission line matrix (TLM) algorithms have been developed intensively in the last decade. Their salient feature is that electromagnetic field is analyzed in the time domain and the samples of relevant physical quantities at nodes located at the discrete points in space are used to represent a physical continuum. These two methods are constantly being improved. The improvements include the application of graded meshes or nonorthogonal cells, application of local approximations or extension of the basic algorithms to the new class of materials such as ferrites or dispersive media. Also, new concepts of space representation of fields have been introduced.

Recognizing the progress achieved in the recent years in the time domain analysis of electromagnetic fields, it should be noted that the explicit algorithms underlaying these methods are not unconditionally stable and the improvements introduced to algorithms affect their stability. Consequently there is a need to investigate the stability criteria for new schemes [6], [9]. In this letter, we shall present ways that the stability of different algorithms can be investigated using functional analysis.

II. Stability Analysis of Explicit Time Domain Algorithms

Let us consider a hyperbolic differential equation

$$\frac{\partial^2}{\partial t^2} f + \mathbf{L} f = 0 \tag{1}$$

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The author is with the Department of Electronics, The Technical University of Gdańsk, Gdańsk, Poland.

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where $\mathbf{L}$ is an elliptic linear differential operator. The hyperbolic equation of this type, supplemented by conditions at $t = 0$ can be solved for $t > 0$ using a classical finite difference explicit algorithm [2]. It is known that the explicit algorithms are conditionally stable. The approach most frequently used to derive the stability condition is known as the von Neumann stability analysis [2], [5]. This analysis involves local expansion of unknown functions into Fourier series and assumes the finite difference representation of the operator. If $\mathbf{L}$ is a negative Laplacian, the von Neumann approach leads the formula known as the Courant-Friedrich-Levy condition. The classical von Neumann analysis can also be expressed in terms of the functional analysis [1], [9].

To investigate the stability of explicit algorithms for the hyperbolic equations it is useful to present a problem in a canonical form:

$$\Delta t^2 \mathbf{R} \frac{\partial^2}{\partial t^2} f + \mathbf{A} f = 0 \tag{2}$$

The time marching algorithm for the above problem is stable if the following conditions are fulfilled [1]:

$$\mathbf{A} = \mathbf{A}^* > 0, \quad \mathbf{R} = \mathbf{R}^* > 0 \tag{3}$$

$$\mathbf{R} - \frac{\mathbf{A}}{4} \geq 0. \tag{4}$$

In other words, for the time-marching algorithm to be stable it is sufficient that both operators $\mathbf{A}$ and $\mathbf{R}$ are self adjoint and positive and, additionally, the operator $\mathbf{R} - 0.25\mathbf{A}$ is nonnegative. The canonical form (2) is obtained from (1) by simply writing it as

$$\frac{\Delta t^2}{\Delta t^2} \mathbf{I} \frac{\partial^2}{\partial t^2} f + \mathbf{L} f = 0 \tag{5}$$

where $\mathbf{I}$ is the identity operator.

Comparing (5) with (2), we get $\mathbf{R} = \mathbf{I}/\Delta t^2$ and $\mathbf{A} = \mathbf{L}$. It can readily be verified that operator $\mathbf{L}$ is symmetric and positive. It suffices to verify the condition (4). This condition is fulfilled when

$$\left| \frac{\mathbf{I}}{\Delta t^2} \right| \geq \frac{\|\mathbf{L}\|}{4} \tag{6}$$

or

$$\Delta t \leq \frac{2}{\sqrt{\|\mathbf{L}\|}}. \tag{7}$$

Thus, the maximal time step in explicit time domain algorithms considered here depends on the norm of the operator $\mathbf{L}$.

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For a self-adjoint bounded operator $\mathbf{L}$ defined in the Hilbert space $\mathcal{H}$ the norm is defined as [3]

$$\|\mathbf{L}\| = \sup_{y \in \mathcal{H}, \|y\|=1} |\langle \mathbf{L}y, y\rangle| = |\lambda_{max}| \tag{8}$$

where λ_{max} is the largest eigenvalue of $\mathbf{L}$.

III. Stability Analysis for One-Dimensional Problem

One important conclusion that follows from the functional analysis approach is that the stability condition depends on how the unknown functions are represented. This is because the norm of the operator depends on the space it acts in. When solving a particular problem, we choose the way the functions are represented and the criteria to measure the accuracy of our solution. This choice is equivalent to the choice of a functional space and affects the norm of the operator, and thus the stability condition. To illustrate this problem in more detail let us consider a one dimensional problem

$$\frac{\partial^2}{\partial t^2} f - b(x)\frac{\partial^2}{\partial x^2} f = 0 \tag{9}$$

$$f(x, t_0) = f_0(x), \quad f(x=0) = f(x=l) = 0 \tag{10}$$

where $b(x) > 0$ is a time independent continuous function of x,

One possible way of solving the above problem is to use a classical finite difference approach, but let us find the solution by means of the method of moments. Let $\mathcal{D}$ denote the domain of operator $\mathbf{L}$ and assume that it allows only functions satisfying Dirichlet conditions at both ends of the interval whose first and second derivatives are both square integrable. By equipping the domain $\mathcal{D}$ with an inner product

$$\langle u, v\rangle = \int_0^t uv\, dx \tag{11}$$

we specify it in terms of the Hilbert space.

It can easily be verified that operator

$$\mathbf{L} = -b(x)\frac{\partial^2}{\partial x^2} \tag{12}$$

is positive and self adjoint. However, if we would like to calculate its norm in this space, we note that the operator $\mathbf{L}$ is unbounded and consequently its norm is infinite. Its norm becomes finite, however, if the operator is allowed to act in a finite dimensional space. This is what happens in practice because we always look for an approximate solution to the problem using a finite number of elements to represent a function. Let us now expand the function $f(x)$ into series of basis functions

$$f(x) = \sum c_i f_i(x) \tag{13}$$

and use the inner product (11) to find the expansion coefficient at any instance of time.

The finite set of basis function defines the approximate finite dimensional subspace of original domain. Consider the following truncated set of basis functions:

$$\sqrt{\frac{2}{l}} \sin\frac{i\pi x}{l} \quad i \le N_M \tag{14}$$

The basis functions (14) span a finite dimensional space $\mathcal{H}_{N_M} \subset \mathcal{D}$ in which the approximate solution is sought. Now it is easy to find the upper bound of the operator.

$$\|\mathbf{L}\| = \|b(x)\frac{\partial^2}{\partial x^2}(\cdot)\| \le \|b_{max}\frac{\partial^2}{\partial x^2}(\cdot)\| = \|\mathbf{L}_m\| \tag{15}$$

where b_{max} is the maximal absolute value of $b(x)$ over the interval $\langle 0, l\rangle$. The eigenvalues λ_i of operator $\mathbf{L}_m$ are given by

$$\lambda_i = \frac{b_{max} i^2 \pi^2}{l^2} \tag{16}$$

and consequently the norm of $\mathbf{L}$

$$\|\mathbf{L}\| \le \frac{b_{max} N_M^2 \pi^2}{l^2}. \tag{17}$$

This leads to the condition

$$\Delta t \le \frac{2l}{\pi N_M \sqrt{b_{max}}}. \tag{18}$$

If we chose an alternative way and represent the function and the operator in a finite difference sense by specifying their values at discrete points, the norm will be changed. If the discretization points are equidistant and the spacing is Δd then [1]

$$\|\mathbf{L}\| \le \frac{4b_{max}}{(\Delta d)^2} \tag{19}$$

yielding the Courant-Friedrich-Levy condition:

$$\Delta t \le \frac{1}{\Delta d\sqrt{b_{max}}} \tag{20}$$

IV. Application to Electromagnetics

The stability analysis described above can be used in electromagnetic problems. Here the operator $\mathbf{L}$ can be specified as

$$\mathbf{L} = \frac{1}{\epsilon_0\mu_0\epsilon(x, y, z)}\nabla \times \frac{1}{\mu(x, y, z)}\nabla \times (\cdot) \tag{21}$$

(other definitions are also possible).

As an example, let us consider a cube Ω with the dimensions $l \times l \times l$. In this region we seek an approximate solution to the hyperbolic equation with an operator defined by (21) given a finite number of expansion functions in the form of normalized products of sines and cosines

$$\sin\frac{i\pi\xi}{l} \quad \text{or} \quad \cos\frac{k\pi\xi}{l} \quad i, k \le N_M. \tag{22}$$

Let $\mathcal{D}$ denote the domain of operator $\mathbf{L}$. The basis functions (22) span a finite dimensional space $\mathcal{H}_{N_M} \subset \mathcal{D}$. We calculate the upper bound of the norm of operator $\|\mathbf{L}\|$. Note that $\|\mathbf{L}\| \le \|\mathbf{L}_m\|$. Where

$$\mathbf{L}_m = (\epsilon_0\mu_0\epsilon_{\min}\mu_{\min})^{-1}\nabla \times \nabla \times (\cdot) \tag{23}$$

and

$$\epsilon_{\min} = \inf \epsilon_r(x, y, z),$$

$$\mu_{\min} = \inf \mu_r(x, y, z) \quad x, y, z \in \Omega. \tag{24}$$

Using the same procedure as for one-dimensional case we find the norm of operator $\mathbf{L}$ in $\mathcal{H}_{N_M}$

$$\|\mathbf{L}\| \leq \|\mathbf{L}_m\| \leq v_{\max}^2 \frac{3N_M^2\pi^2}{l} \tag{25}$$

where $v_{\max} = (\epsilon_0\mu_0\epsilon_{\min}\mu_{\min})^{-1/2}$ is the maximum velocity for a plane wave in the structure. Using the above estimation we get the following stability condition

$$\Delta t \leq \frac{2l}{v_{\max} N_M \pi\sqrt{3}}. \tag{26}$$

If the region Ω is a rectangular prism with the dimensions $a \times b \times l$ and the upper bound for i, l, m in the trigonometric expansion functions is K_M, L_M, N_M, then the condition (26) becomes

$$\Delta t \leq \frac{2}{v_{\max}\pi\sqrt{\left(\frac{K_M}{a}\right)^2 + \left(\frac{L_M}{b}\right)^2 + \left(\frac{N_M}{l}\right)^2}}. \tag{27}$$

Discretizing the space Ω in the z direction with the step Δd and using $K_M \times L_M$ sine and cosine basis functions to in the x and y directions, the stability condition derived using (19) is

$$\Delta t \leq \frac{2}{v_{\max}\sqrt{\left(\frac{\pi K_M}{a}\right)^2 + \left(\frac{\pi L_M}{b}\right)^2 + \left(\frac{2}{\Delta d}\right)^2}}. \tag{28}$$

For the discretization of all three coordinates with steps Δx, Δy, Δd, we shall get the well-known Courant condition [5]

$$\Delta t \leq \frac{1}{v_{\max}\sqrt{\left(\frac{1}{\Delta x}\right)^2 + \left(\frac{1}{\Delta y}\right)^2 + \left(\frac{1}{\Delta d}\right)^2}}. \tag{29}$$

At this point it is interesting to observe that the derivation described above provides stability criteria for a few recently published time domain algorithms. For instance, (27) and (28) are the stability criteria for the Total Eigenfunction Expansion and Partial Eigenfunction Equations schemes derived in [8] (for sine and cosine basis functions). In a compact 2-D/FDTD algorithm described in [4] and investigated subsequently by Cangellaris [6], the functions are represented by samples at uniformly discretized cartesian coordinates x, y and the variation in the z direction given in the form $\exp(-j\beta z)$. For this case, the functional analysis approach gives

$$\Delta t \leq \frac{1}{v_{\max}\sqrt{\left(\frac{1}{\Delta x}\right)^2 + \left(\frac{1}{\Delta y}\right)^2 + \left(\frac{\beta}{2}\right)^2}}. \tag{30}$$

This condition is identical as the one given in [4] and [6]. Also, the stability of a hybrid spectral/FDTD method introduced recently by Cangellaris *et al.* [7] follows from condition (7).

V. Conclusions

The application of the functional analysis to the investigation of the stability of time domain algorithms has been presented. It was shown that the method can easily be applied to the investigation of the properties of novel time domain schemes for Maxwell's equations.

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